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The diffusion of a viscous vortex ring in a rotating fluid

By J. A. JOHNSON
School of Mathematics and Physics, University of East Anglia, Norwich

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A viscous vortex ring is considered as an initial disturbance to a uniformly rotating fluid. Inertial waves are generated by the disturbance and propagate through the fluid affecting the diffusion of the vortex ring with time t. The rate of diffusion is found to be proportional to $t^{\frac{2}{3}}$. This result is compared with the diffusion rates for non-rotating fluids and for conducting fluids under a uniform magnetic field.

1. Introduction

When a small disturbance is introduced into an unbounded fluid in uniform motion, it will appear as a type of vortex ring provided the influence of the disturbance is confined to a small region of the fluid. For then the vortex lines associated with the disturbance must be closed and confined to the region of the disturbance. Thus in order to investigate the spreading of a small disturbance through a uniform rotating fluid, it is reasonable to choose a viscous vortex ring at the origin as the initial distribution of disturbance vorticity. As the uniform rotation imposes a constraint on motions in the fluid, it seems unlikely that the vortex ring will remain coherent as it spreads. Our interest in this paper is in the maximum extent of the disturbance after time t, and not in the smaller details of the motion.

Phillips (1956) has shown that the decay of viscous vortex rings may be used to represent the motion in the final period of decay of any turbulent motion. He found that the extent of the vortex ring increases with time t as $\{\nu\surd(t-t_0)\}$, where ν is the kinematic viscosity and t_0 is a virtual time origin. Saffman (1961) has extended this result to include conducting fluids. He showed that the final stage of decay of a localized disturbance in a uniform magnetic field may be represented by two viscous vortex rings, whose centres are moving in opposite directions along the magnetic field with the Alfvén wave velocity a. That is, if a viscous vortex ring disturbs a conducting fluid under a uniform magnetic field, the rate of growth of the disturbance along the magnetic field is at.

In a uniformly rotating viscous fluid, an initial disturbance will produce inertial waves that propagate through the fluid. The properties of such waves have been discussed in detail by Chandrasekhar (1961) and by Phillips (1963). As this problem in a rotating fluid is not tractable using the methods of Phillips or Saffman, a new approach had to be devised to determine a length scale that measured the extent of the disturbance at time t. The results obtained for the rotating fluid are unusual and unexpected, and are given in § 2. In order to verify that the method used gives a valid length scale, the same approach is applied to a non-rotating fluid and to a conducting fluid under a uniform magnetic field, and correct solutions are obtained. Finally, §5 provides a discussion of the differences between the diffusion rates for the different fluids.

2. Rotating fluid

The vorticity equation for an incompressible fluid, expressed in a frame of reference rotating with angular velocity $\mathbf{\Omega}$, is

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\mathbf{u}.\nabla\boldsymbol{\omega} = (\boldsymbol{\omega}+2\mathbf{\Omega}).\nabla\mathbf{u}+\nu\nabla^2\boldsymbol{\omega}. \tag{2.1}$$

In this equation, $\boldsymbol{\omega}(\mathbf{x},t)$ represents the vorticity field and $\mathbf{u}(\mathbf{x},t)$ the velocity field, both measured in the rotating frame. If the Rossby number $U/\Omega l$ and the Reynolds number Ul/ν are both small, where l and U are basic length and velocity scales, the non-linear terms may be neglected compared with the Coriolis term and the viscous term. This assumption requires that variations of the vorticity $\boldsymbol{\omega}$ shall be small compared with the mean vorticity $2\mathbf{\Omega}$, and then equation (2.1) reduces to

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\boldsymbol{\omega} = 2\mathbf{\Omega}.\nabla\mathbf{u}. \tag{2.2}$$

The application of the operator

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2$$

to equation (2.2) gives a single equation for $\boldsymbol{\omega}$:

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)^2\nabla^2\boldsymbol{\omega}+(2\mathbf{\Omega}.\nabla)^2\boldsymbol{\omega} = 0. \tag{2.3}$$

This represents a type of wave equation for $\boldsymbol{\omega}$ and is the equation of inertial waves.

Solutions of equation (2.3) may be found by using the Fourier transforms

$$\boldsymbol{\omega}_i(\mathbf{x},t) = (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{k}.\mathbf{x}}\chi_i(\mathbf{k},t)\,d\mathbf{k}, \tag{2.4a}$$

$$\mathbf{u}_i(\mathbf{x},t) = (2\pi)^{-\frac{3}{2}}\int e^{i\mathbf{k}.\mathbf{x}}\phi_i(\mathbf{k},t)\,d\mathbf{k}, \tag{2.4b}$$

where the suffix i indicates the ith component, $d\mathbf{k} = dk_1dk_2dk_3$ and the integrals are over all wave-number space. The Fourier transform of the ith component of (2.3) is

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)^2 k^2\chi_i+4(\mathbf{\Omega}.\mathbf{k})^2\chi_i = 0. \tag{2.5}$$

The general solution of equation (2.5) is

$$\chi_i(\mathbf{k},t) = B_ie^{-\nu k^2t}\cos\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right)+C_ie^{-\nu k^2t}\sin\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right), \tag{2.6}$$

where $B_i(\mathbf{k})$ and $C_i(\mathbf{k})$ are arbitrary functions. The solution for the transform ϕ_i may be obtained from the Fourier transform of (2.2),

$$\frac{\partial \chi_i}{t} + \nu k^2 \chi_i = 2i\mathbf{\Omega}.\mathbf{k}\phi_i.$$

Hence
$$ik\phi_i = C_i e^{-\nu k^2 t}\cos\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right) - B_i e^{-\nu k^2 t}\sin\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right). \tag{2.7}$$

A further relationship between $\boldsymbol{\omega}$ and $\mathbf{u}$ is

$$\nabla \times \boldsymbol{\omega} = -\nabla^2 \mathbf{u},$$

which has a Fourier transform whose ith component is

$$k^2\phi_i = i\epsilon_{imn}k_m\chi_n. \tag{2.8}$$

Thus we can reduce the number of arbitrary functions B_i, C_i from six to three. Substituting (2.6) and (2.7) into equation (2.8) gives

$$k\left\{B_i \sin\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right) - C_i\cos\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right)\right\}$$
$$= \epsilon_{imn}k_m\left\{B_n\cos\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right) + C_n\sin\left(\frac{2\mathbf{\Omega}.\mathbf{k}}{k}t\right)\right\}.$$

As B_i and C_i are independent of t, we must have

$$kB_i = \epsilon_{imn}k_m C_n, \tag{2.9a}$$
$$-kC_i = \epsilon_{imn}k_m B_n. \tag{2.9b}$$

These two equations show that the vectors $\mathbf{B}$ and $\mathbf{C}$ are in different orthogonal directions normal to the wave-number vector. Therefore $\mathbf{k}.\boldsymbol{\chi} = 0$ and the solution (2.6) satisfies the continuity equation for an incompressible fluid. The remaining three arbitrary functions are determined from the initial distribution of vorticity that disturbs the uniform motion.

The viscous vortex ring

The viscous vortex ring may be represented at time $t = 0$ by the distribution of vorticity defined by

$$\boldsymbol{\omega}(\mathbf{x}, 0) = l^{-5}\mathbf{A}\times\mathbf{x}\,e^{-x^2/4l^2}, \tag{2.10}$$

where the vector $\mathbf{A}$ is the axis of the ring. The total momentum of the vortex ring is proportional to $\rho\mathbf{A}$, where ρ is the density of the fluid. The vorticity has a maximum value at $x = l\sqrt{2}$. If the radius of the vortex ring is defined as the radius of the circle of maximum vorticity, then $l\sqrt{2}$ is the radius of the ring at time $t = 0$. The streamlines of the velocity field corresponding to (2.10) are shown in figure 1 in the plane containing $\mathbf{A}$ and the rotation vector $\mathbf{\Omega}$. For simplification, the co-ordinate axes have been chosen so that $\mathbf{\Omega} = (0, 0, \Omega)$. The vortex lines are circles with centres on $\mathbf{A}$, lying in planes perpendicular to $\mathbf{A}$. The vector $\mathbf{A}$ gives the orientation and initial total momentum of the vortex ring.

The Fourier transform of (2.10) is the initial condition imposed on the transform of the vorticity, that is

$$\chi_i(\mathbf{k},0) = (2\pi)^{-\frac{3}{2}} \int e^{-i\mathbf{k}.\mathbf{x}} l^{-5} \epsilon_{ipq} A_p x_q e^{-x^2/4l^2} d\mathbf{x}$$

$$= -4\sqrt{2}\, i \epsilon_{ipq} A_p k_q e^{-k^2 l^2}$$

$$= B_i, \qquad (2.11)$$

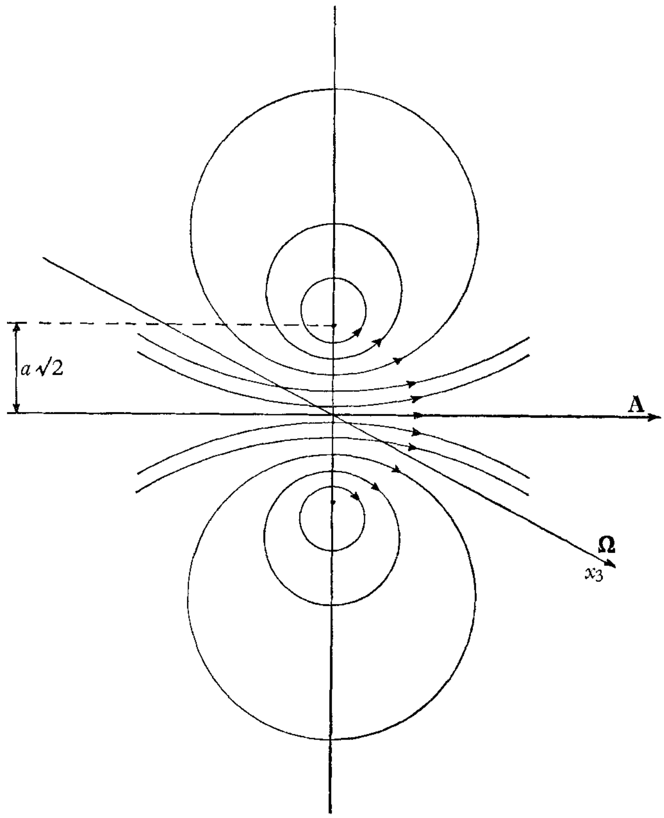

FIGURE 1. Streamlines of vortex ring at time $t = 0$.

from equation (2.6). Substituting (2.11) into equation (2.9a) gives

$$kC_i = 4\sqrt{2}\, i(k_m A_i k_m - k_m A_m k_i)\, e^{-k^2 l^2}.$$

Therefore, eliminating B_i and C_i from (2.6), we have

$$\chi_i = -4\sqrt{2}\, i\, e^{-k^2(l^2+\nu t)} \left\{ \epsilon_{ipq} A_p k_q \cos \frac{2\Omega k_3}{k} t - \frac{1}{k}(k^2 A_i - \mathbf{k}.\mathbf{A} k_i) \sin \frac{2\Omega k_3}{k} t \right\}. \qquad (2.12)$$

This is the transform of the vorticity field for the viscous vortex ring after time t has elapsed. The coefficients $\mathbf{B}$ and $\mathbf{C}$ are proportional to $\mathbf{k} \times \mathbf{A}$ and $\mathbf{k} \times (\mathbf{A} \times \mathbf{k})/k$ respectively and are therefore mutually perpendicular and normal to the wave-number vector in accordance with (2.9).

The solution for large Ωt

The solution (2.12) for the transform χ_i decreases to zero as time increases and the initial disturbance will be dissipated by viscosity. However, it is difficult to determine more precisely the behaviour of the disturbance for $\Omega t \gg 1$. We are reduced to looking for integrals of χ_i that may be evaluated for large Ωt. Saffman's method cannot be used to invert (2.12) as the factor $2\Omega/k$ makes the integration intractable.

If Parseval's theorem is applied to the vorticity ω_i and its transform χ_i, then

$$\int \omega_i^2(\mathbf{x})\,d\mathbf{x} = -\int \chi_i^2(\mathbf{k})\,d\mathbf{k}, \tag{2.13}$$

as $\chi_i(-\mathbf{k}) = -\chi_i(\mathbf{k})$. Substituting for χ_1, in (2.13) and integrating, we obtain a polynomial in inverse powers of Ωt, whose dominant term for large Ωt is given by

$$\int \omega_1^2(\mathbf{x})\,d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}}{5(l^2+\nu t)^{\frac{5}{2}}}[(4A_1^2+3A_2^2+3A_3^2)+O\{(\Omega t)^{-1}\}]. \tag{2.14}$$

Similar integrals are obtained for ω_2^2 and ω_3^2. The sum of these three integrals gives

$$\int \omega^2(\mathbf{x})\,dx = \frac{2(2\pi)^{\frac{3}{2}}A^2}{(l^2+\nu t)^{\frac{5}{2}}}.$$

This integral is exact as all terms involving Ωt cancel out. It represents the total vorticity of the fluid and is a measure of the rate of dissipation of the disturbance vorticity by viscosity.

A second integral, that is also an even function of $\boldsymbol{\omega}$, may be obtained from the differential of the inverse transform of (2.4*a*). As

$$\chi_i(\mathbf{k},t) = (2\pi)^{-\frac{3}{2}}\int e^{-i\mathbf{k}.\mathbf{x}}\,\omega_i(\mathbf{x},t)\,d\mathbf{x},$$

its first derivative is

$$\frac{d\chi_i}{dk_j} = -i(2\pi)^{-\frac{3}{2}}\int e^{-i\mathbf{k}.\mathbf{x}}\,x_j\,\omega_i\,d\mathbf{x}.$$

Hence, using Parseval's theorem again,

$$\int x_j^2\omega_i^2\,d\mathbf{x} = -\int (d\chi_i/dk_j)^2\,d\mathbf{k}. \tag{2.15}$$

Substituting for $d\chi_1/dk_1$ in (2.15), we obtain

$$\int x_1^2\omega_1^2 d\mathbf{x} = \frac{16\,(2\pi)^{\frac{3}{2}}}{315\,(l^2+\nu t)^{\frac{3}{2}}}\left[(4A_1^2+5A_2^2+3A_3^2)\,(\Omega t)^2+\ldots+\frac{\lambda \sin 4\Omega t}{(\Omega t)^7}\right],$$

where λ is a constant. The evaluation of this integral, although tedious, is straightforward and the solution contains a finite number of terms. Therefore, for large Ωt, the second integral for ω_1 is

$$\int x_1^2\omega_1^2 d\mathbf{x} = \frac{16(2\pi)^{\frac{3}{2}}}{315\,(l^2+\nu t)^{\frac{3}{2}}}[(4A_1^2+5A_2^2+3A_3^2)\,(\Omega t)^2+O(\Omega t)], \tag{2.16}$$

which is strongly dependent on the basic rotation of the fluid. As a small disturbance far from the origin is weighted by a large x_1-co-ordinate, the integral gives an indication of the rate of diffusion in the x_1-direction of the vorticity of the disturbance away from the initial concentration near the origin.

Although no physical explanation has been found for the asymmetry of the components of $\mathbf{A}$ in (2.16), it is connected with the constraint imposed on the fluid by the uniform rotation about the x_3-axis. The integral of $x_2^2\omega_2^2$ has the factor $(5A_1^2+4A_2^2+3A_3^2)$, whereas the integral of $x_3^2\omega_3^2$ has $8(5A_1^2+5A_2^2+8A_3^2)$. These numerical differences ultimately produce faster diffusion along the axis of rotation compared with directions normal to this axis.

From the integrals (2.14) and (2.16), we may define a length scale which is proportional to the dimensions of the disturbance after time t. We define the length L_1 by

$$L_1^2 = \int x_1^2\omega_1^2 d\mathbf{x} \bigg/ \int \omega_1^2 d\mathbf{x}.$$

Therefore for large Ωt,

$$L_1 \sim F(\mathbf{A})\,(\Omega t)\,(l^2+\nu t)^{\frac{1}{2}}, \tag{2.17}$$

where $F(\mathbf{A})$ is a function of the direction of $\mathbf{A}$. L_1 may be considered as a characteristic length of the disturbance measured along the x_1-axis. Moreover, the rate of diffusion in the x_1-direction is measured by the rate of increase of the length L_1. Two other length scales L_2 and L_3 may be defined from similar integrals and they will determine the rate of diffusion in the x_2- and x_3-directions respectively. In general, for large Ωt, the length scales extend at the rate $(\Omega t)\,(l^2+\nu t)^{\frac{1}{2}}$, but the rate of extension is fastest along the x_3-axis, parallel to the rotation vector. The extent of the disturbance is an ellipsoid whose major axis is along the axis of rotation.

For an inviscid fluid, the rate of spreading of vorticity is only $l(\Omega t)$; this is the rate at which inertial waves spread.

The solution for small Ωt

For $t \ll \Omega^{-1}$, we may replace $\sin 2\Omega k_3 t/k$ by $2\Omega k_3 t/k$ in (2.12) and then the integrals for ω_1 simplify. For small Ωt, we have

$$\int \omega_1^2(\mathbf{x})\,d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}}{(l^2+\nu t)^{\frac{5}{2}}}\{(A_2^2+A_3^2)+O(\Omega t)\}, \tag{2.18}$$

$$\int x_1^2\omega_1^2(\mathbf{x})\,d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}}{(l^2+\nu t)^{\frac{3}{2}}}\{(A_2^2+A_3^2)+O(\Omega t)\}. \tag{2.19}$$

Therefore, for small Ωt,

$$L_1 \sim (l^2+\nu t)^{\frac{1}{2}}.$$

Before commenting further on these results, we shall derive the corresponding expressions for a non-rotating fluid and for a conducting fluid under a uniform magnetic field. Then the effects produced by the rotation will be seen more clearly.

3. Non-rotating fluid

In order to obtain a direct comparison with §2, the same method will be used, but now equation (2.2) simplifies to

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\boldsymbol{\omega} = 0,$$

the normal vorticity equation. Its Fourier transform is

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\chi(\mathbf{k}, t) = 0, \tag{3.1}$$

which has a general solution of the form

$$\chi = \mathbf{B}(\mathbf{k})\, e^{-\nu k^2 t}.$$

The same distribution of vorticity is introduced at time $t = 0$ in the form of a viscous vortex ring. The Fourier transform of this distribution is given by equation (2.11), and, as

$$B_i(\mathbf{k}) = \chi_i(\mathbf{k}, 0),$$

the solution of (3.1) becomes

$$\chi_i(\mathbf{k}, t) = -4\sqrt{2}\, i\, \epsilon_{ipq} A_p k_q e^{-k^2(l^2+\nu t)}.$$

Substituting for χ_1 in (2.13), the first integral of ω_1 is

$$\int \omega_1^2 d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}}{(l^2+\nu t)^{\frac{5}{2}}}(A_2^2 + A_3^2). \tag{3.2}$$

This integral is exact and valid for all values of t. The second integral is given by (2.15),

$$\int x_1^2 \omega_1^2 d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}}{(l^2+\nu t)^{\frac{3}{2}}}(A_2^2 + A_3^2), \tag{3.3}$$

which is also valid for all t.

The length scale of the disturbance, defined as before from the ratio of the above two integrals, is

$$L_1 = (l^2 + \nu t)^{\frac{1}{2}}, \tag{3.4}$$

and all three length scales L_1, L_2, L_3 are equal. It is to be expected that in a non-rotating fluid, the extension of length scales will be isotropic and that there will be no favoured direction. Equally, in a rotating fluid, one expects to find a preferred direction, as illustrated by the results of the previous section. We may verify that the chosen length scale is reasonable by comparing it with a result obtained by Phillips (1956). Using an entirely different method, he showed that in a non-rotating fluid the radius of a viscous vortex ring increased with time as $\{\nu(t-t_0)\}^{\frac{1}{2}}$ where t_0 is a virtual time origin. The result is equivalent to (3.4) as $(\nu t_0)^{\frac{1}{2}}$ is a natural length scale of the problem.

The integrals (3.2) and (3.3) are identical with integrals (2.18) and (2.19) for small Ωt. This means that when $t \ll O(\Omega^{-1})$, the rotation has no effect on the behaviour of the disturbance, and the diffusion rates are the same as for a non-rotating fluid. Rotational effects only become important after a time of order Ω^{-1} has elapsed.

4. Conducting fluid

Using the rotation of Saffman (1961), the equations of motion in e.m.u. for a conducting viscous fluid are

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+\mathbf{u}.\nabla\boldsymbol{\omega} = \boldsymbol{\omega}.\nabla\mathbf{u}+\nu\nabla^2\boldsymbol{\omega}+\frac{1}{\rho}\nabla\times(\mathbf{j}\times\mathbf{B}), \tag{4.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{H})+\lambda\nabla^2\mathbf{H}, \tag{4.2}$$

$$\mathbf{j} = \frac{1}{4\pi}\nabla\times\mathbf{H}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \nabla.\mathbf{H} = 0, \tag{4.3}$$

where $\mathbf{B}, \mathbf{H}, \mathbf{j}$ are the magnetic induction, the magnetic intensity and the current. The constants ρ, λ, μ are density, magnetic diffusivity and permeability.

If a uniform magnetic field $\mathbf{H}_0 = H_0\mathbf{n}$ where $\mathbf{n}$ is a unit vector is applied to a conducting fluid which is slightly disturbed by the introduction of a vortex ring at the origin, the perturbed magnetic field may be written as

$$\mathbf{H} = \mathbf{H}_0+H_0\mathbf{h},$$

where $\mathbf{h}$ is small. Now, using (4.3),

$$\frac{1}{\rho}\nabla\times(\mathbf{j}\times\mathbf{B}) = \frac{\mu H_0}{4\pi\rho}\nabla\times\{(\nabla\times\mathbf{h})\times\mathbf{H}_0\}+O(h^2)$$

$$= \frac{\mu H_0^2}{4\pi\rho}(\mathbf{n}.\nabla)\nabla\times\mathbf{h}+O(h^2). \tag{4.4}$$

Similarly,

$$\nabla\times(\mathbf{u}\times\mathbf{H}) = H_0(\mathbf{n}.\nabla)\mathbf{u}-H_0(\mathbf{h}.\nabla)\mathbf{u}-H_0(\mathbf{u}.\nabla)\mathbf{h}. \tag{4.5}$$

Therefore, substituting (4.5) into (4.2)

$$H_0\frac{\partial \mathbf{h}}{\partial t} = H_0(\mathbf{n}.\nabla)\mathbf{u}-H_0(\mathbf{h}.\nabla)\mathbf{u}-H_0(\mathbf{u}.\nabla)\mathbf{h}+\lambda H_0\nabla^2\mathbf{h}. \tag{4.6}$$

If the magnetic Reynolds number $R_m = Ul/\lambda$ is small and if $h = O(R_m)$, the non-linear terms may be neglected and equation (4.6) reduces to

$$\left(\frac{\partial}{\partial t}-\lambda\nabla^2\right)\mathbf{h} = (\mathbf{n}.\nabla)\mathbf{u}. \tag{4.7}$$

Similarly, if the Reynolds numbers Ul/ν is small, the vorticity equation (4.1) with (4.4) becomes

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\boldsymbol{\omega} = a^2(\mathbf{n}.\nabla)\nabla\times\mathbf{h}, \tag{4.8}$$

where $a = (\mu H_0^2/4\pi\rho)^{\frac{1}{2}}$ is the Alfvén wave velocity. Eliminating $\mathbf{h}$ from (4.8) and the curl of (4.7), a single equation is obtained for the vorticity $\boldsymbol{\omega}$,

$$\left(\frac{\partial}{\partial t}-\lambda\nabla^2\right)\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\boldsymbol{\omega}-a^2(\mathbf{n}.\nabla)^2\boldsymbol{\omega} = 0. \tag{4.9}$$

The Fourier transform of the ith component of (4.9) is

$$\left(\frac{\partial}{\partial t}+\lambda k^2\right)\left(\frac{\partial}{\partial t}+\nu k^2\right)\chi_i+a^2(\mathbf{n}.\mathbf{k})^2\chi_i = 0, \tag{4.10}$$

where χ_i is defined by equation (2.4a).

Saffman has shown that the solution for $\lambda \neq \nu$ eventually, for large enough t, has a similar form to the solution for $\lambda = \nu$. Therefore, for simplicity, we here assume that $\lambda = \nu$, and the general solution of (4.10) becomes

$$\chi_i(\mathbf{k},t) = e^{-\nu k^2 t}\{B_i\cos ak_1 t+C_i\sin ak_1 t\}, \tag{4.11}$$

where the co-ordinate axes have been chosen so that $\mathbf{H}_0 = (H_0, 0, 0)$. The initial disturbance to the conducting fluid is the viscous vortex ring defined by (2.10). Hence, by (2.11) and (4.11),

$$\chi_i(\mathbf{k},0) = B_i = -4\sqrt{2}i\epsilon_{ipq}A_p k_q e^{-k^2 l^2}.$$

If $\mathbf{J}$ is the Fourier transform of the current $\mathbf{j}$, then the transforms of equations (4.3) and (4.8) give

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\chi_i = 4\pi k_1 a^2 J_i.$$

Hence,
$$J_i(\mathbf{k},t) = \frac{1}{4\pi a}e^{-\nu k^2 t}(C_i\cos ak_1 t-B_i\sin ak_1 t),$$

and, if the current is initially zero, then

$$J_i(\mathbf{k},0) = \frac{1}{4\pi a}C_i = 0.$$

Equation (4.11) with $C_i = 0$ gives the velocity transform as

$$\phi_i = 4\sqrt{2}\,e^{-k^2(l^2+\nu t)}(\delta_{ij}-k_i k_j/k^2)A_j\cos ak_1 t,$$

which has the same form as equation (22) in Saffman (1961). The solution for χ_i reduces to

$$\chi_i(\mathbf{k},t) = -4\sqrt{2}i\epsilon_{ipq}A_p k_q\cos(ak_1 t)\,e^{-k^2(l^2+\nu t)},$$

and the integrals for ω_1, corresponding to (2.14) and (2.16) for the rotating fluid, may be evaluated. For the conducting fluid,

$$\int\omega_1^2 d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}(A_2^2+A_3^2)}{2(l^2+\nu t)^{\frac{5}{2}}}\left\{1+\exp\left(-\frac{a^2t^2}{2(l^2+\nu t)}\right)\right\}, \tag{4.12}$$

$$\int x_1^2\omega_1^2 d\mathbf{x} = \frac{(2\pi)^{\frac{3}{2}}(A_2^2+A_3^2)}{2(l^2+\nu t)^{\frac{3}{2}}}\left\{\frac{a^2t^2}{l^2+\nu t}+1+\exp\left(-\frac{a^2t^2}{2(l^2+\nu t)}\right)\right\}. \tag{4.13}$$

The length scale of the disturbance, defined as before from the ratio of the above two integrals, is

$$L_1 \sim at \quad \text{for} \quad a^2t^2 \gg l^2+\nu t, \tag{4.14}$$

and
$$L_1 \sim (l^2+\nu t)^{\frac{1}{2}} \quad \text{for} \quad a^2t^2 \ll l^2+\nu t. \tag{4.15}$$

This is the length scale measured in the direction of the magnetic field. By similar calculations, it is possible to show that

$$L_2 = L_3 = (l^2+\nu t)^{\frac{1}{2}}$$

for all t. Thus the results obtained are in accordance with the solution derived by Saffman which showed that the initial vortex ring splits up into two vortex rings which move in opposite directions along the uniform magnetic field with the Alfvén wave velocity. For small t, the viscous diffusion is the dominant effect and the diffusion rate is $t^{\frac{1}{2}}$ in all directions. However, for large t, the distance between the rings is $2at$ and the diffusion rate is at along the magnetic field but remains at $t^{\frac{1}{2}}$ in other directions.

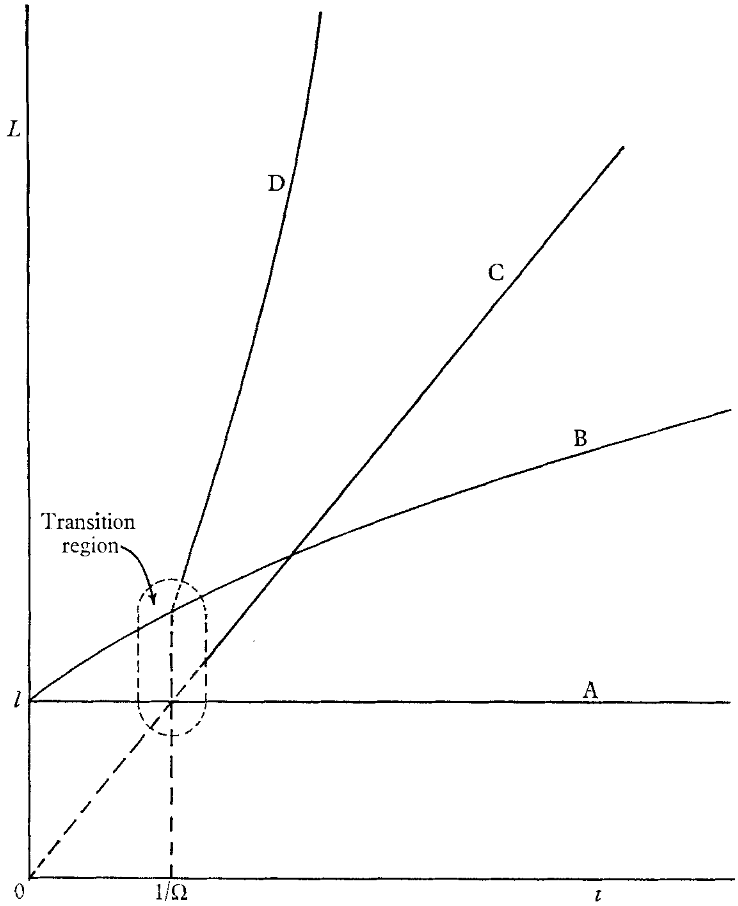

FIGURE 2. The extension of the length scale L for different fluids. Curve A, inviscid, non-rotating, $L = l$; curve B, viscous, non-rotating, $L \sim (l^2 + \nu t)^{\frac{1}{2}}$; curve C, inviscid, rotating, $L \sim l(\Omega t)$; curve D, viscous, rotating, $L \sim (\Omega t)\,(l^2 + \nu t)^{\frac{1}{2}}$.

5. Discussion

In the preceding two sections we have shown that our method for calculating the extent of the disturbance gives satisfactory results for a non-rotating fluid and for a conducting fluid in a uniform magnetic field. This suggests that our approach is generally valid and that the unexpected diffusion rates obtained for the rotating fluid are correct. Although we may expect that the magnetic field and the rotation would have similar effects on the fluid as they both imposed a restraint on the fluid that produces a preferred direction, our theory suggests otherwise.

A critical difference between the influence of the magnetic field and of the uniform rotation is clearly shown by the group velocities of the waves they generate. The Alfvén-wave group velocity is **a**, constant in magnitude, and directed along the uniform magnetic field. Whereas, the inertial wave group velocity is

$$\mathbf{c}_g = \frac{2}{k^3}(\mathbf{\Omega} \times \mathbf{k}) \times \mathbf{k},$$

whose magnitude varies with the wave-number, and its direction lies in the plane of $\mathbf{\Omega}$ and $\mathbf{k}$ but is always perpendicular to $\mathbf{k}$. The factors $2\Omega/k$ and a appear in the sinusoidal terms of (2.12) and (4.11), the expressions for the Fourier transforms of the vorticity, and it is the presence of the k that prohibits the use of Saffman's method for the rotating fluid case. The net result is that energy is propagated in all directions by the inertial waves but only along the magnetic field by the Alfvén waves. Moreover, the length scales L_1, L_2, L_3 are all proportional to $t^{\frac{3}{2}}$ in the rotating fluid, whereas only the length scale L_1 along the magnetic field is proportional to t in the conducting fluid.

For the rotating fluid, the basic results are summarized in figure 2. For small Ωt, the diffusion rates are the same as for a non-rotating fluid, that is $L \sim (l^2+\nu t)^{\frac{1}{2}}$. However, for large Ωt, the rate of diffusion accelerates to become

$$L \sim (\Omega t)\,(l^2+\nu t)^{\frac{1}{2}}.$$

Thus a transition period exists when $\Omega t = O(1)$ during which the diffusion process becomes affected by the uniform rotation. In an inviscid fluid, the disturbance is spread entirely by inertial waves, whose group velocity has magnitude $O(\Omega/k)$, and we have shown that, when $t > O(\Omega^{-1})$, the length scale increases linearly with time. It seems that more detailed knowledge of the flow pattern for large Ωt is required before sufficient insight into the problem can be obtained to give a satisfactory physical reason that will explain this unusual behaviour. At present it is only possible to say that the accelerated diffusion rate is due to a combination of viscous diffusion and the peculiar properties of inertial waves.

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